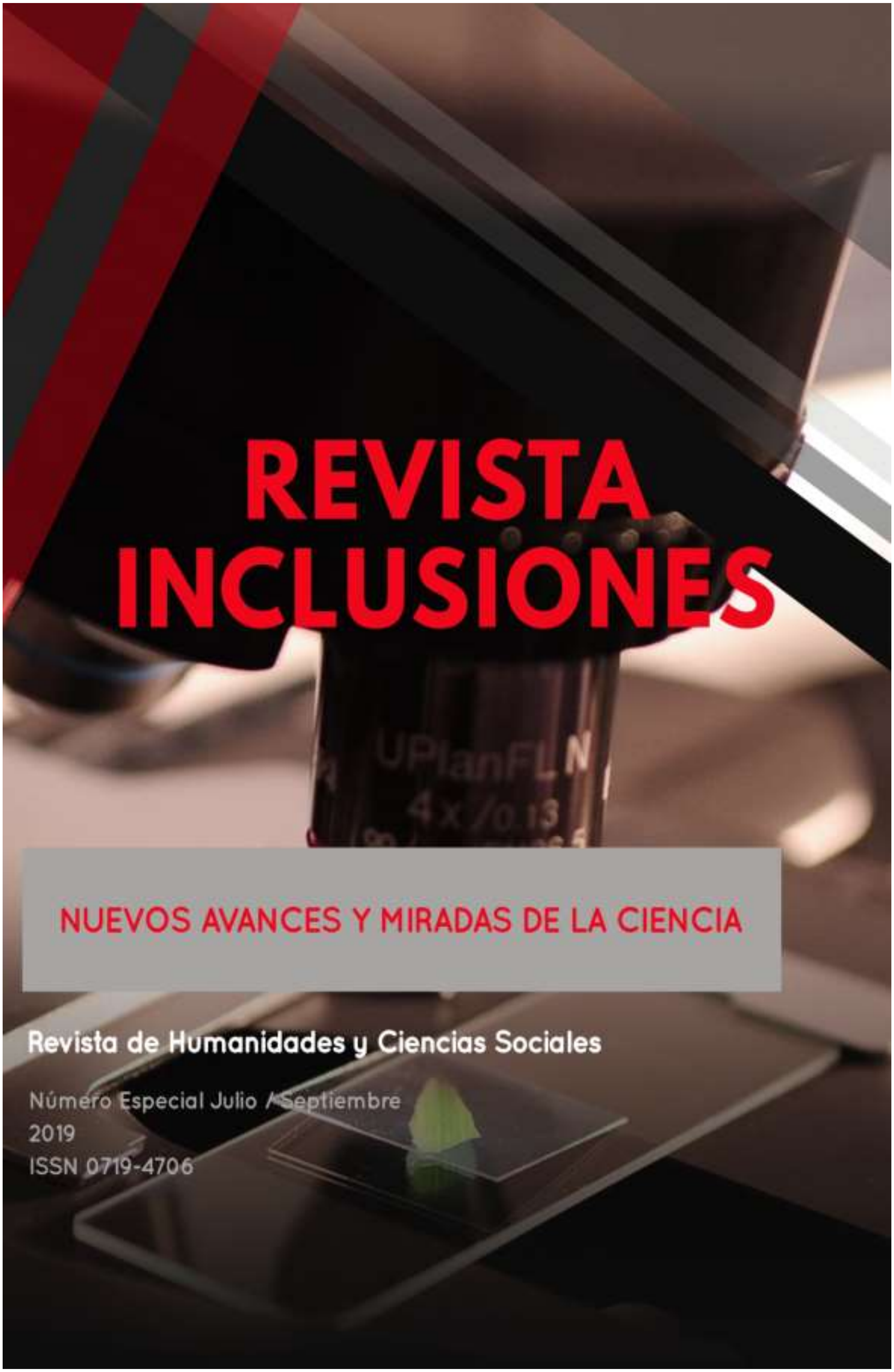
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NEW DETERMINATION OF THE FRACTURAL DERIVATIVE OF ITS PROPERTIES

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Abstract

The article aims to present a new definition of the fractional derivative of a function by replacing the exponential function with an increasing logarithmic function. In addition to the introduced definition, proof of relevance and the need to introduce a new fractional derivative, the properties of the fractional derivative and their proof are considered. All the properties of the introduced fractional derivative are proved, similar to the properties of the ordinary derivative. Also, fractional derivatives in the indicated sense are calculated for some elementary functions. The article discusses fractional derivative theorems necessary for solving fractional integral and differential equations.

Keywords

Fractional derivative – Riemann-Liouville definition – Caputo definition
Properties of fractional derivative

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Introduction

The solution of fractional differential and fractional integral equations is impossible without an initial study of the concepts of "fractional derivative" and "fractional integral". The fractional derivative is as old as calculus. In 1695 L. Hospital asked what it means $\frac{d^n t}{dx^n}$, if an $n = \frac{1}{2}$. Since then, a large number of researchers have tried to define the fractional derivative.

The problem of the study, which the article is devoted to, is to provide an accurate definition of the fractional derivative and evidence of its properties, which can be used in fractional differential and integral calculus.

The urgency of the problem posed is that at this stage of the development of the science of fractional integration and differentiation, only some of the provisions and equations have been investigated, among which the Abel equation, the Riemann-Liouville equation, and the Caputo equation are distinguished.

The definition of Riemann-Liouville suggests that for $\alpha \in [n-1; n)$ α-derivative of the function f calculated by the formula[1]

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

According to the definition of Caputo, for$\alpha \in [n-1; n)$ α-the derivative of the function *f* is calculated by the formula[2]

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}} dx.$$

The definitions described above satisfy the linearity property of the fractional derivative. This is the only property inherited from the first derivative in all definitions. However, here are some of the failures of these definitions:

1.- The definition of the Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$ ($(D_a^{\alpha}(1)) = 0$ for derivative Caputo) if α is not a natural number[3];
2.- All fractional derivatives do not satisfy the well-known derivative formula of the product of two functions $D_a^{\alpha}(fg) = f D_a^{\alpha}(g) + g D_a^{\alpha}(f)$;

[1] J. He; X. Zhang; L. Liu; Y. Wu and Y. Cui, "Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions". Bound Value Problems. 2018 y K. Oldham and J. Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitrary Order (USA: Academic Press, 2014).
[2] A. Kilbas; H. Srivastava and J. Trujillo, Theory and Applications of Fractional Differential Equations (North-Holland New York: In Mmath Studies, 2016) y I. Podlubny, Fractional Differential Equations (USA: Academic Press, 2017).
[3] R. P. Agarwal; M. Benchohra and S. Hamani, "A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions". Acta. Appl. Math. 2014; D. Shayakhmetova & A. Chaklikova, "Development of the intermediator of intercultural communication based on public argumentative speech", Opción, Vol: 34 num 85-2 (2018): 149-185 y W. Jiang and N. Kosmatov, Existence results for a functional boundary value problem of fractional differential equations. Bound Value Problems. 2018.

3.- All fractional derivatives do not satisfy the well-known formula for the derivative of the quotient of two functions $D_a^\alpha\left(\frac{f}{g}\right)=\frac{gD_a^\alpha(f)-fD_a^\alpha(g)}{g^2}$;
4.- All fractional derivatives do not satisfy the chain rule $D_a^\alpha(f^\circ g)(t)=f^\alpha\big(g(t)\big)g^\alpha(t)$[4];
5.- All fractional derivatives do not satisfy $D^\alpha D^\beta f=D^{\alpha+\beta}f$ generally[5];

The definition of Caputo suggests that the function is differentiable.

Proceeding from this, we decided to introduce a new definition of a fractional derivative, and also to prove its main properties - in turn, the proof of the properties of the fractional derivative will show the validity and correctness of the introduced definition[6].

Methods

In the work were used some empirical and theoretical research methods, which include:

- The method of transition from the general to the particular, which allows one to study not the fractional derivative formula as such, but to show the adequacy of the introduction of certain terms, as well as the correctness and accuracy of the introduction of the new definition;
- The method of systematization and classification, by which the necessary basic information is selected to derive a new definition, to construct an approximation of the fractional integral function, and also to select a theoretical material;
- The partial-search method, consisting in the search for scientific works, articles, dissertations, manuals, their translation into Russian.

Findings

Before introducing the definition, we point out that the fractional derivative of the function *f (t)* of order *D* in our work is denoted as $T_D(f)(t)$, a fractional integral – $I_D(f)(t)$.

Definition. Set function $f:[a;b]\to R$. Then the corresponding fractional derivative of the function f of order *D* is defined as

$$T_D(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-f(t)}{\varepsilon},$$

[4] K. S. Miller, An Introduction to Fractional Calculus and Fractional Differential Equations (New York: J. Wiley and Sons, 2013).
[5] A. Yousef, "A new definition of fractional derivative", Journal of Computational and Applied Mathematics (2014): 64-72.
[6] M. P. Ardakani; A. Lashkarian & M. Sadeghzadeh, "The Translatability/Untranslatability of Poetics: Eliot's" Ash Wednesday" and its two Persian translations", UCT Journal of Social Sciences and Humanities Research, Vol: 3 num 1 (2015): 52- 60.

For all $t > 0$, $t \neq 1$. If f is -differentiable on some interval $[a;b], a > 0, \lim_{t\to 0^+} f^{(D)}(t)$ exists, then$f^{(D)}(0) = \lim_{t\to 0^+} f^{(D)}(t)$.

We will sometimes write $T_D(f)(t)$ instead $f^{(D)}(t)$ to denote the corresponding fractional derivatives of a function *f* of order *D*. Moreover, if the corresponding fractional derivative of a function *f* of order *D* exists, then we simply say that *f* is -differentiable.

It should be noted that $T_D(t^p) = \frac{pt^{p-1}}{ln\frac{c}{t}}$.

As a consequence of the above definition, we obtain the following useful theorem.

Theorem 1. If the function $f:[a;b] \to R$ is a D-differentiated at $t_0 > 0,$ then f is continuous on t_0.

Evidence. Because *of* $f = \left(t_0 + \frac{\varepsilon}{ln\frac{c}{t_0}}\right) - f(t_0) = \frac{f\left(t_0+\frac{\varepsilon}{ln\frac{c}{t_0}}\right)-f(t_0)}{\varepsilon}$, then

$$\lim_{\varepsilon\to 0}\left[\left(t_0 + \varepsilon\frac{\varepsilon}{ln\frac{c}{t_0}}\right) - f(t_0)\right] = \lim_{\varepsilon\to 0}\frac{f\left(t_0+\frac{\varepsilon}{ln\frac{c}{t_0}}\right)-f(t_0)}{\varepsilon}\cdot\lim_{\varepsilon\to 0}\varepsilon.$$

Make a replacement. Now let $\frac{\varepsilon}{ln\frac{c}{t_0}} = h,$ then

$$\lim_{h\to 0}[(t_0 + h) - f(t_0)] = f^{(D)}(t_0)\cdot 0$$

This implies that

$$\lim_{h\to 0} f(t_0 + h) = f(t_0).$$

Therefore, the function *f* is continuous on t_0.

It is easy to show that T_D satisfies all the properties of the following theorem.

Theorem 2. Let *f, g* be -differentiable at a point$t > 0$.

Then the following properties are executed:

1. $T_D(af + bg) = aT_D(f) + bT_D(g), a, b \in R$;
2. $T_D(t^p) = \frac{pt^{p-1}}{ln\frac{c}{t}}, p \in R$;
3 $T_D\left(\frac{f}{g}\right) = \frac{gT_D(f)-fT_D(g)}{g^2}$;
4. $T_D(fg) = fT_D(g) + gT_D(f)$;
5. $T_D(\lambda) = 0$ for all permanent functions $f(t) = \lambda$.

6. In addition, if the function *f* is differentiable, then

7. $T_D(f)(t) = \frac{1}{ln\frac{c}{t}}\frac{df}{dt}(t)$

Proof 1 property: $T_D(af+bg)=aT_D(f)+bT_D(g), a, b \in R$.

Let be $aT_D(f)=\lim_{\varepsilon\to 0} a\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-f(t)}{\varepsilon}=a\lim_{\varepsilon\to 0}\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-f(t)}{\varepsilon}$. We introduce a replacement $\frac{\varepsilon}{ln\frac{c}{t}}=h$, will get $aT_D(f)=a\lim\limits_{h\to 0}[(t+h)-f(t)]=af^{(D)}(t)\cdot 0=af(t)$.

Similarly, we obtain for the second term $bT_D(g)=\lim_{\varepsilon\to 0} b\frac{g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-g(t)}{\varepsilon}=b\lim_{\varepsilon\to 0}\frac{g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-g(t)}{\varepsilon}$. We introduce a replacement $\frac{\varepsilon}{ln\frac{c}{t}}=h$, will get $bT_D(g)=b\lim\limits_{h\to 0}[(t+h)-g(t)]=bg^{(D)}(t)\cdot 0=bg(t)$.
Add up the results, we obtain that $aT_D(f)+bT_D(g)=af(t)+bg(t)$.

On the other hand, $T_D(af+bg)=\lim_{\varepsilon\to 0}\left[a\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-f(t)}{\varepsilon}+b\frac{g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-g(t)}{\varepsilon}\right]$. We introduce a replacement $\frac{\varepsilon}{ln\frac{c}{t}}=h$, will get $T_D(af+bg)=\lim_{\varepsilon\to 0}[a((t+h)-f(t))+b((t+h)-g(t))]$. We can break this expression into the sum of two limits and make the constants in each of them beyond the limit sign: $T_D(af+bg)=a\lim\limits_{h\to 0}[(t+h)-f(t)]+b\lim\limits_{h\to 0}[(t+h)-g(t)]$. Consider every term $T_D(af+bg)=af(t)+bg(t)$. Compare the left and right sides:

$$af(t)+bg(t)=af(t)+bg(t).$$

Proof 2 properties: $T_D(t^p)=\frac{pt^{p-1}}{ln\frac{c}{t}}, p\in R$.

There are two ways to prove this property. The first case is suitable only for t> 0, the second case is suitable for all t. We first consider the first case, which involves the introduction of a logarithmic function: $t^p=y$. Take the logarithm of base e:

$$t^p=y;$$
$$lny=lnt^p;$$
$$lny=plnt.$$

Find the derivative of this implicitly given function:

$$(l\acute{n}y)=(p\acute{l}nt);$$
$$\frac{1}{y}\acute{y}=p\frac{1}{t};$$
$$\acute{y}=p\frac{y}{t}=p\frac{t^p}{t}=pt^{p-1}.$$

Here we find a derivative of order 1, but in our case, it is necessary to find a derivative of order *D*, and for all *t*.

The second case is considered for all *t* and uses the Newton binomial formula.

$$T_D(t^p) = \lim_{\varepsilon\to 0} \frac{f\left(t + \frac{\varepsilon}{ln\frac{c}{t}}\right)^p - f(t)^p}{\varepsilon}.$$

According to the Newton binomial formula:

$$\left(t + \frac{\varepsilon}{ln\frac{c}{t}}\right)^p - t^p =$$
$$= C_p^0 t^p + C_p^1 t^{p-1}\frac{\varepsilon}{ln\frac{c}{t}} + C_p^2 t^{p-2}\left(\frac{\varepsilon}{ln\frac{c}{t}}\right)^2 + \cdots + C_p^p\left(\frac{\varepsilon}{ln\frac{c}{t}}\right)^p - t^p.$$

We substitute this expression into the definition of a fractional derivative and, given ε, we get:

$$\lim_{\varepsilon\to 0}\frac{C_p^1 t^{p-1}\frac{\varepsilon}{ln\frac{c}{t}} + 0 + 0 + \cdots 0}{\varepsilon} = \frac{1}{ln\frac{c}{t}}\frac{p!}{1!\,(p-1)!}t^{p-1} = \frac{pt^{p-1}}{ln\frac{c}{t}}.$$

Proof 3 properties: $T_D\left(\frac{f}{g}\right) = \frac{gT_D(f) - fT_D(g)}{g^2}$.

We write the definition of a fractional derivative with the expression $\frac{f}{g}$:

$$T_D\left(\frac{f}{g}\right) = \lim_{\varepsilon\to 0}\frac{\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)}{g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)} - \frac{f(t)}{g(t)}}{\varepsilon}.$$

Let's look at the common denominator and write in one fraction:

$$T_D\left(\frac{f}{g}\right) = \lim_{\varepsilon\to 0}\frac{\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)}{g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)} - \frac{f(t)}{g(t)}}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)g(t) - f(t)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)}{g(t)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)\varepsilon}.$$

Add and subtract the product of functions $f(t)g(t)$, we will receive:

$$T_D\left(\frac{f}{g}\right) = \lim_{\varepsilon\to 0}\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)g(t) - f(t)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right) - f(t)g(t) + f(t)g(t)}{g(t)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)\varepsilon} =$$

$$= \lim_{\varepsilon\to 0}\frac{g(t)\left[f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-f(t)\right]-f(t)\left[g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-g(t)\right]}{g(t)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)\varepsilon}.$$

To get the fractional derivative formula in the numerator, multiply and divide by each term:

$$T_D\left(\frac{f}{g}\right)=\lim_{\varepsilon\to 0}\frac{\frac{g(t)\left[f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-f(t)\right]}{\varepsilon}\varepsilon-\frac{f(t)\left[g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-g(t)\right]}{\varepsilon}\varepsilon}{g(t)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)\varepsilon}=$$

$$=\lim_{\varepsilon\to 0}\frac{g(t)T_D(f)(t)-f(t)T_D(g)(t)}{g(t)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)\varepsilon}=\frac{g(t)T_D(f)(t)-f(t)T_D(g)(t)}{g^2(t)}.$$

Proof 4 properties: $T_D(fg)=fT_D(g)+gT_D(f)$.

We write the definition of a fractional derivative for the expression fg:

$$T_D(fg)=\lim_{\varepsilon\to 0}\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-f(t)g(t)}{\varepsilon}.$$

Add and subtract the product of functions $f(t)g\frac{\varepsilon}{ln\frac{c}{t}}$, will get:

$$T_D(fg)=$$

$$=\lim_{\varepsilon\to 0}\frac{f\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-f(t)g(t)-f(t)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)+f(t)g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)}{\varepsilon}.$$

Group and divide by two limits:

$$T_D(fg)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon\frac{\varepsilon}{ln\frac{c}{t}}\right)-f(t)}{\varepsilon}g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)+$$

$$+f(t)\lim_{\varepsilon\to 0}\frac{g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)-g(t)}{\varepsilon}.$$

Note that in the first term is the definition of the fractional derivative of the function $f(t)$, in the second term is the definition of the fractional derivative of the function $g(t)$, those:

$$T_D(f)(t)\lim_{\varepsilon\to 0}g\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)+f(t)T_D(g)(t)=T_D(f)(t)g(t)+f(t)T_D(g)(t).$$

Proof 5 properties: $T_D(ƛ) = 0$ for all permanent functions$f(t) = ƛ$.

Let be $ƛ = const$, then from the definition of the fractional derivative of the function follows:

$$T_D(ƛ) = \lim_{\varepsilon\to 0}\frac{f\left(ƛ + \frac{\varepsilon}{ln\frac{c}{t}}ƛ\right) - f(ƛ)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{f(ƛ) - f(ƛ)}{\varepsilon} = 0$$

Since the numerator is not a function that tends to zero, but the number 0 is obtained, the limit will be equal to 0.

Proof of 6 property: f is differentiable, then

$$T_D(f)(t) = \frac{1}{ln\frac{c}{t}}\cdot\frac{df}{dt}(t).$$

To prove this part of the theorem, let us imagine that $\frac{\varepsilon}{ln\frac{c}{t}} = h$ in the first definition and let $\varepsilon = \frac{h}{ln\frac{c}{t}}$ in the first definition and let

$$T_D(f)(t) = \lim_{\varepsilon\to 0}\frac{f\left(t + \frac{\varepsilon}{ln\frac{c}{t}}\right) - f(t)}{\varepsilon} = \lim_{h\to 0}\frac{f(t+h) - f(t)}{\frac{h}{ln\frac{c}{t}}} =$$
$$= \frac{1}{ln\frac{c}{t}}\lim_{h\to 0}\frac{f(t+h) - f(t)}{h} = \frac{1}{ln\frac{c}{t}}\cdot\frac{df}{dt}(t).$$

Based on the above definition, it is possible to derive fractional derivatives of some functions that are tabular when working with a conventional derivative.

The corresponding fractional derivative of some functions will be as follows[7]

1. $T_D(t^p) = \frac{pt^{p-1}}{ln\frac{c}{t}}, p \in R;$
2. $T_D(1) = 0;$
3. $T_D(e^{at}) = a\frac{e^{at}}{ln\frac{c}{t}}, a \in R;$
4. $T_D(sinbt) = b\frac{cosbt}{ln\frac{c}{t}}, b \in R;$
5. $T_D(cosbt) = -b\frac{sinbt}{ln\frac{c}{t}}, b \in R;$

To show the fidelity of our replacements and the fidelity of the properties introduced above based on the fractional derivative, we will prove each of these properties. It should be noted that the first and second properties have already been proved above; therefore, we consider the proofs of properties 3-6.

[7] J. He; X. Zhang; L. Liu; Y. Wu and Y. Cui, “Existence and asymptotic analysis of positive… y I. Podlubny, Fractional Differential Equations…

Proof 3 properties: $T_D(e^{at}) = a\frac{e^{at}}{ln\frac{c}{t}}, a \in R$.

From the definition of the fractional derivative of the function, we have:

$$T_D(e^{at}) = \lim_{\varepsilon\to 0}\frac{e^{a\left(t+\frac{\varepsilon}{ln\frac{c}{t}}\right)} - e^{at}}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{e^{at+a\frac{\varepsilon}{ln\frac{c}{t}}} - e^{at}}{\varepsilon}.$$

We use property 6 for the -differentiable function *f*, that is, $T_D(f)(t) = \frac{1}{ln\frac{c}{t}}\frac{df}{dt}(t)$. We introduce a replacement $a\frac{\varepsilon}{ln\frac{c}{t}} = h$ let it go $\varepsilon = a\frac{h}{ln\frac{c}{t}}$, then the definition of the fractional derivative for e^{at} written in the following form:

$$T_D(e^{at}) = \lim_{\varepsilon\to 0}\frac{e^{at+a\frac{\varepsilon}{ln\frac{c}{t}}} - e^{at}}{\varepsilon} = \lim_{h\to 0}\frac{e^{at+h} - e^{at}}{a\frac{h}{ln\frac{c}{t}}} = \frac{a}{ln\frac{c}{t}}\lim_{h\to 0}\frac{e^{at+h} - e^{at}}{h}.$$

Consider the logarithm property *blnc* = *lnc* ^{b} and apply it to the definition of the limit, we get:

$$e^{at+h} - e^{at} = e^{at}\cdot e^{h} - e^{at} = e^{at}(e^{h} - 1);$$
$$T_D(e^{at}) = \frac{a}{ln\frac{c}{t}}\lim_{h\to 0}\frac{e^{at}(e^{h} - 1)}{h} = \frac{a}{ln\frac{c}{t}}e^{at}\lim_{h\to 0}\frac{e^{h} - 1}{h}.$$

We introduce the following replacement: $e^{h} - 1 = y$.

Then $e^{h} = y + 1, h = \ln(y + 1)$.

Given the fact that the exhibitor is continuous $\lim_{h\to 0} e^{h} = e^{0} = 1$, and knowing that $h \to 0, y \to 0$, will get:

$$T_D(e^{at}) = \frac{a}{ln\frac{c}{t}}e^{at}\lim_{h\to 0}\frac{e^{h} - 1}{\varepsilon} = \frac{a}{ln\frac{c}{t}}e^{at}\lim_{y\to 0}\frac{y}{ln(y + 1)}.$$

Make another substitution $\frac{1}{y} = n, y = \frac{1}{n}$. With $y \to 0, n \to \infty$ we get the following:

$$T_D(e^{at}) = \frac{a}{ln\frac{c}{t}}e^{at}\lim_{y\to 0}\frac{y}{\ln(y + 1)} = \frac{a}{ln\frac{c}{t}}e^{at}\lim_{n\to\infty}\frac{1}{nln\left(\frac{1}{n} + 1\right)}.$$

Apply to this equality the logarithm property *blnc* = *lnc* ^{b}, then we get:

$$nln\left(\frac{1}{n} + 1\right) = ln\left(\frac{1}{n} + 1\right)^{n} = ln\left(1 + \frac{1}{n}\right)^{n}.$$

Based on the obtained equality, we rewrite the definition of a fractional derivative:

$$T_D(e^{at}) = \frac{a}{ln\frac{c}{t}}e^{at}\lim_{n\to\infty}\frac{1}{nln\left(\frac{1}{n} + 1\right)} = \frac{a}{ln\frac{c}{t}}e^{at}\lim_{n\to\infty}\frac{1}{ln\left(1 + \frac{1}{n}\right)^{n}} =$$

$$= \frac{a}{ln\frac{c}{t}} e^{at} \frac{1}{\lim\limits_{n\to\infty} ln\left(1+\frac{1}{n}\right)^n}$$

Apply property to the resulting expression $\lim_{x\to x_0} \ln(f(t)) = \ln\left(\lim\limits_{x\to x_0} f(t)\right)$, here $f(t) -$ some function that has a limit $\lim\limits_{x\to x_0} f(t)$, and this limit is positive. Given the equality, we find that with the existence of a positive limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and its continuity, the limit is written as:

$$\lim_{n\to\infty} ln\left(1+\frac{1}{n}\right)^n = ln \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = lne = 1.$$

Considering the fact that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e -$ Considering the fact that:

$$T_D(e^{at}) = \frac{a}{ln\frac{c}{t}} e^{at} \lim_{n\to\infty} \frac{1}{ln\left(1+\frac{1}{n}\right)^n} = \frac{a}{ln\frac{c}{t}} e^{at} \frac{1}{lne} = \frac{a}{ln\frac{c}{t}} e^{at}$$

Proof 4 properties: $T_D(sinbt) = b\frac{cosbt}{ln\frac{c}{t}}, b \in R$.

Consider the derivative of the sine, taking into account the definition of the fractional derivative:

$$T_D(sinbt) = \lim_{\varepsilon\to 0} \frac{sin\left(bt + b\frac{\varepsilon}{ln\frac{c}{t}}\right) - sin(bt)}{\varepsilon}.$$

We take into account the above theorem then $T_D(f)(t) = \frac{1}{ln\frac{c}{t}} \cdot \frac{df}{dt}(t)$. Let be $b\frac{\varepsilon}{ln\frac{c}{t}} = h$ let it go $\varepsilon = b\frac{h}{ln\frac{c}{t}}$. Therefore, the definition of the fractional derivative of sinbt will be as follows:

$$T_D(sinbt) = \lim_{\varepsilon\to 0} \frac{sin\left(bt + b\frac{\varepsilon}{ln\frac{c}{t}}\right) - sin(bt)}{\varepsilon} =$$
$$= \lim_{h\to 0} \frac{sin(bt+h) - sin(bt)}{b\frac{h}{ln\frac{c}{t}}} = \frac{b}{ln\frac{c}{t}} \lim_{h\to 0} \frac{sin(bt+h) - sin(bt)}{h}.$$

Using the sinus difference formula, we get:

$$T_D(sinbt) = \frac{b}{ln\frac{c}{t}} \lim_{h\to 0} \frac{sin(bt+h) - sin(bt)}{h} =$$

$$= \frac{b}{ln\frac{c}{t}} \lim_{h\to 0} \frac{2sin\frac{(bt+h-bt)}{2}\cdot cos\frac{(bt+h+bt)}{2}}{h} =$$

$$= \frac{b}{ln\frac{c}{t}} \lim_{h\to 0} \frac{sin\frac{h}{2}\cdot \cos(bt+\frac{h}{2})}{\frac{\varepsilon bh}{2}} = \frac{b}{ln\frac{c}{t}} \cos\left(bt+\frac{0}{2}\right) \lim_{h\to 0} \frac{sin\frac{h}{2}}{\frac{h}{2}} = \frac{b}{ln\frac{c}{t}} cosbt$$

It should be noted that $\lim_{h\to 0} \frac{sin\frac{h}{2}}{\frac{h}{2}} = 1$, since this is the first wonderful limit.

From here,

$$T_D(sinbt) = \frac{b}{ln\frac{c}{t}} cosbt = b\frac{cosbt}{ln\frac{c}{t}}.$$

Similarly, we prove the 5th property.

Proof 5 properties: $T_D(cosbt) = -b\frac{sinbt}{ln\frac{c}{t}}, b \in R.$

We write the definition of the fractional derivative for the function *cosbt*, we get:

$$T_D(cosbt) = \lim_{\varepsilon\to 0} \frac{cos\left(bt + b\frac{\varepsilon}{ln\frac{c}{t}}\right) - cos(bt)}{\varepsilon}.$$

We introduce the following replacement $b\frac{\varepsilon}{ln\frac{c}{t}} = h$ let it go $\varepsilon = b\frac{h}{ln\frac{c}{t}}$. Then we get the following:

$$T_D(cosbt) = \lim_{\varepsilon\to 0} \frac{cos\left(bt + b\frac{\varepsilon}{ln\frac{c}{t}}\right) - cos(bt)}{\varepsilon} =$$

$$= \lim_{h\to 0} \frac{cos(bt+h) - cos(bt)}{b\frac{h}{ln\frac{c}{t}}} = \frac{b}{ln\frac{c}{t}} \lim_{h\to 0} \frac{cos(bt+h) - cos(bt)}{h}.$$

By the formula of the difference of cosines we get:

$$T_D(cosbt) = \frac{b}{ln\frac{c}{t}} \lim_{h\to 0} \frac{cos(bt+h) - cos(bt)}{h} =$$

$$= \frac{b}{ln\frac{c}{t}} \lim_{h\to 0} \frac{-2sin\frac{(bt+h+bt)}{2}\cdot sin\frac{(bt+h-bt)}{2}}{h} =$$

$$= \frac{b}{ln\frac{c}{t}} \lim_{h\to 0} \frac{-2sin\left(bt+\frac{h}{2}\right) sin\frac{h}{2}}{\frac{h}{2}} = -\frac{b}{ln\frac{c}{t}} \sin\left(bt+\frac{0}{2}\right) \lim_{h\to 0} \frac{sin\frac{h}{2}}{\frac{h}{2}} =$$

$$= -\frac{b}{ln\frac{c}{t}} sinbt = -b\frac{sinbt}{ln\frac{c}{t}}.$$

It should be noted that $\lim_{h\to 0}\frac{sin\frac{h}{2}}{\frac{h}{2}} = 1$ since this is the first wonderful limit.

It is well known that if $\acute{f}(t)$ limited to $I = [a; b]$, then *f* is uniformly continuous on *I*. However, the reverse should not be true. To see this, consider $f(t) = 2\sqrt{t}$ on $I = [0; 1]$. Then the function *f* is uniformly continuous on [0; 1], but $\acute{f}(t)$ not limited here. However, the limitations $f^{(D)}(t)$ and the continuity of the function *f* on *I* (the continuity of the function *f* in 0 in the subspace of topology is equivalent to the right continuity of the function *f* in 0), which, by the above proposal, ensures the uniform continuity of the function *f* on *I*.

Conclusions

Based on the above definition, we can conclude that the introduced fractional derivative is a special case of derivatives that are used in more scientific knowledge and practical problem solving, but our definition, obtained by replacing the exponential function with an increasing logarithmic one, allows more convenient way to solve fractional differential equations and derive the definition of a fractional integral.

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